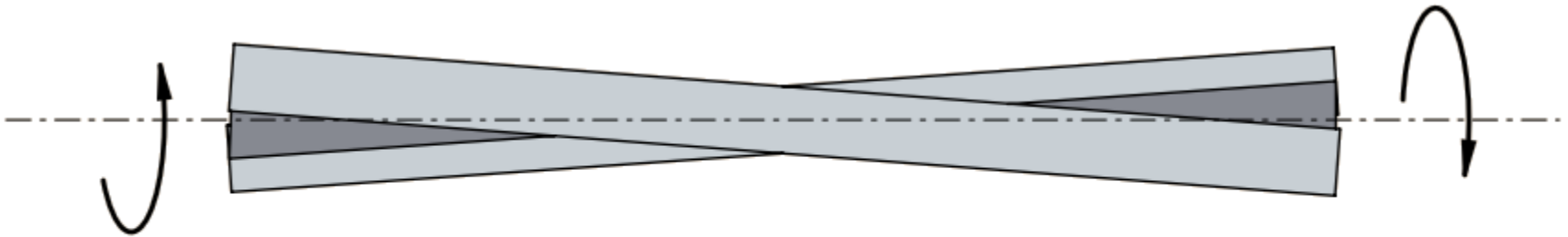


CIENKOŚCIENNE KONSTRUKCJE METALOWE

Wykład 8: Swobodne i nieswobodne skręcanie cienkościennych elementów konstrukcyjnych o przekroju otwartym

SKRĘCANIE SWOBODNE (CZYSSTE)

Występuje, gdy na końce pręta działają momenty skręcające o przeciwnych znakach. Wówczas deplanacja nie natrafia na opór, tzn. występuje swoboda paczenia się przekrojów.



Ends free to warp

Kąt skręcenia wynosi

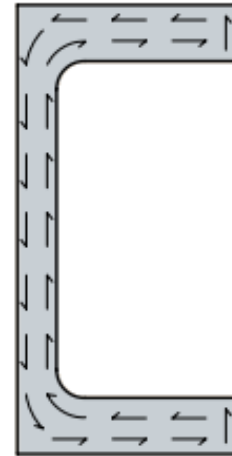
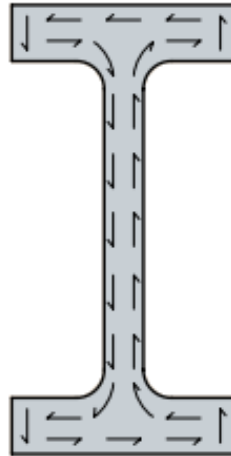
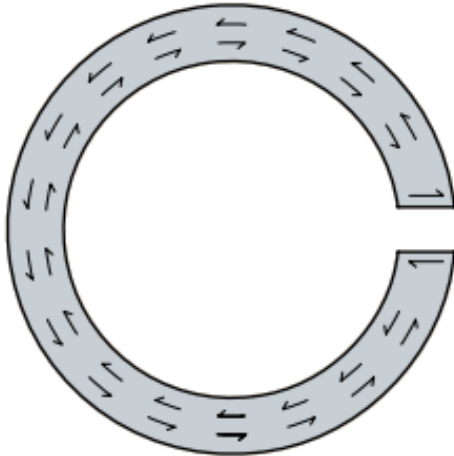
$$\varphi = \frac{M_x l}{GI_T}$$

jednostkowy kąt skręcenia

$$\varphi' = \frac{M_x}{GI_T}$$

SKRĘCANIE SWOBODNE (CZYSTE)

Strumień naprężeń



Stress variation
(in all elements)

Shear stresses in open sections

$$\tau = G t_i \varphi'$$

lub

$$\tau = \frac{M_x}{I_T} t_i$$

gdzie t_i jest grubością ścianki przekroju

MOMENT BEZWŁADNOŚCI CZYSTEGO SKRĘCANIA

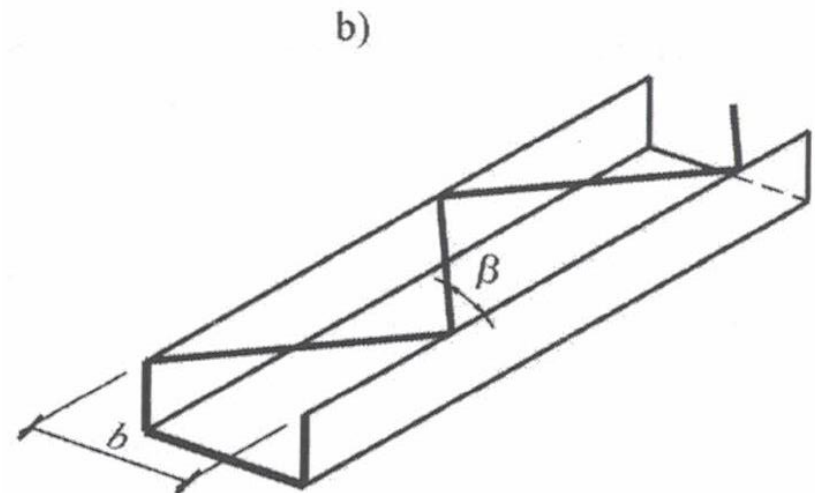
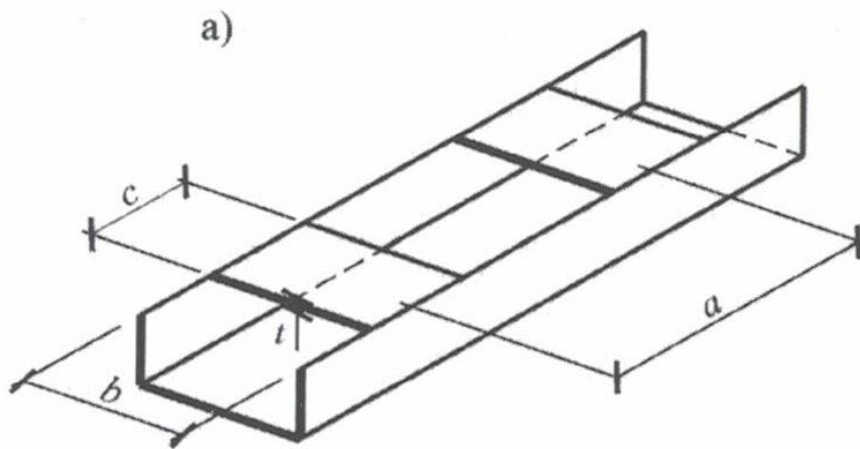
$$I_T = \frac{\alpha}{3} \sum_{i=1}^m b_i t_i$$

Rodzaj kształtownika	Współczynniki α według literatury		
	[144]	[118]	[11]
Kątownik walcowany równoramienny	0,86÷1,10	1,00	1,00
Ceownik walcowany	0,98÷1,25	1,12	1,12
Dwuteownik walcowany zwykły	1,16÷1,44	1,20	1,30
Dwuteownik walcowany szerokostopowy	1,21÷1,47		
Teownik walcowany	0,92÷1,25		1,15
Zetownik walcowany	1,13÷1,20		
Spawany dwuteownik z usztywniającymi przeponami przymocowanymi do środka i półek		1,50	
Spawany teownik z trójkątnymi przeponami przymocowanymi do środka i półek		1,40	
Nitowany dwuteownik bez pasów podłużnych		0,25	
Nitowany dwuteownik z pasami podłużnymi		0,50	

MOMENT BEZWŁADNOŚCI CZYSTEGO SKRĘCANIA

$$I_{Tu} = \frac{4A_z^2}{ab} \left(\frac{b^2 G}{12EI_p} + \frac{1,2}{A_p} \right)^{-1}$$

$$I_{Tu} = \frac{4A_z^2 EA_k}{Gb^2} \sin^2 \beta \cos^2 \beta$$



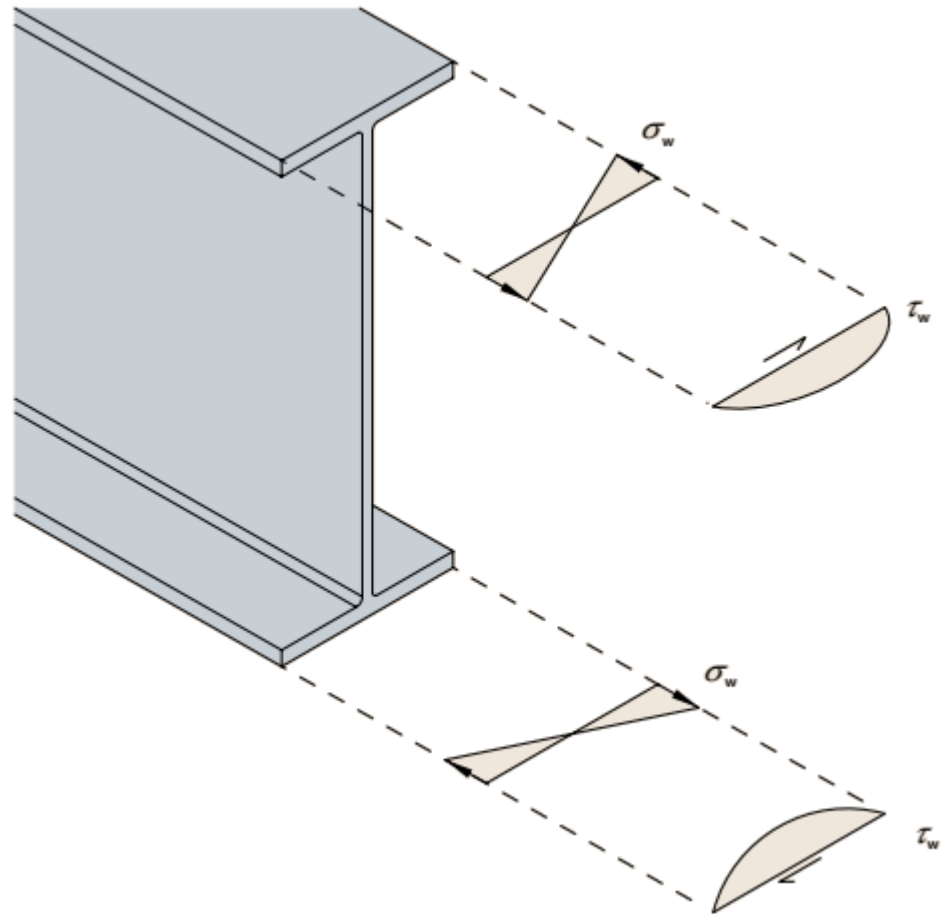
SKRĘCANIE NIESWOBODNE

Ponieważ dwa oddalone od siebie przekroje dx nie mają możliwości deplanacji, następuje odkształcenie wzdłuż włókien x . Przemieszczenia i odkształcenia wynoszą odpowiednio:

$$u(x, s) = -\varphi'(x)\omega(s),$$
$$\varepsilon_x = \frac{\partial u}{\partial x} = -\varphi''\omega.$$

Powstają wówczas normalne naprężenia wycinkowe

$$\sigma_\omega = E\varepsilon_x = -E\varphi''\omega$$



WPLÝW SKRĘCANIA NA KSZTAŁTOWNIKI

SECTION TYPE	SHAPE	ST VENANT	WARPING
Circular hollow sections	○	✓	—
Rectangular and elliptical hollow sections	□ □ ○	✓	✗
Angles, Tees and cruciform sections	⊥ L +	✓	✗
Twin-flanged rolled and fabricated sections	I I C	✓	✓
Thin cold-formed sections	Z Σ	✗	✓

Key: ✓ = significant; ✗ = negligible; — = does not act

R-NIE RÓŻNICZKOWE PROBLEMU

Równanie kątów skręcenia pręta nieswobodnie skręcanego

$$EI_{\omega}\varphi^{(4)} - GI_T\varphi^{(2)} = m_x(x).$$

Można je wyrazić jako

$$\varphi^{(4)} - k^2\varphi^{(2)} = \frac{m_x(x)}{EI_{\omega}},$$

gdzie

$$k^2 = \frac{GI_T}{EI_{\omega}}.$$

SIŁY PRZEKROJOWE

Moment czystego skręcania

$$M_T(x) = GI_T \varphi^{(1)}(x),$$

bimoment

$$B(x) = -EI_\omega \varphi^{(2)},$$

moment giętno-skrętny

$$M_\omega(x) = -EI_\omega \varphi^3(x),$$

moment skręcający

$$M(x) = M_T + M_\omega = GI_T \varphi^{(1)}(x) - EI_\omega \varphi^3(x).$$

WARUNKI PODPARCIA

W wyniku całkowanie r-nia otrzymuje się cztery stałe całkowania, które należy wyznaczyć z warunków brzegowych na obu końcach prętów.

- zablokowanie skręcenia i deplanacji

$$\varphi = 0, \quad \varphi^{(1)} = 0,$$

- zablokowane skręcenie i możliwa deplanacja

$$\varphi = 0, \quad B = -EI_{\omega}\varphi^{(2)} = 0 \rightarrow \varphi^{(2)} = 0,$$

- swoboda skręcenia i możliwa deplanacja

$$\varphi^{(1)} = 0, \quad M = GI_T\varphi^{(1)} - EI_{\omega}\varphi^{(3)} = M_x \rightarrow \varphi^{(3)} = 0,$$

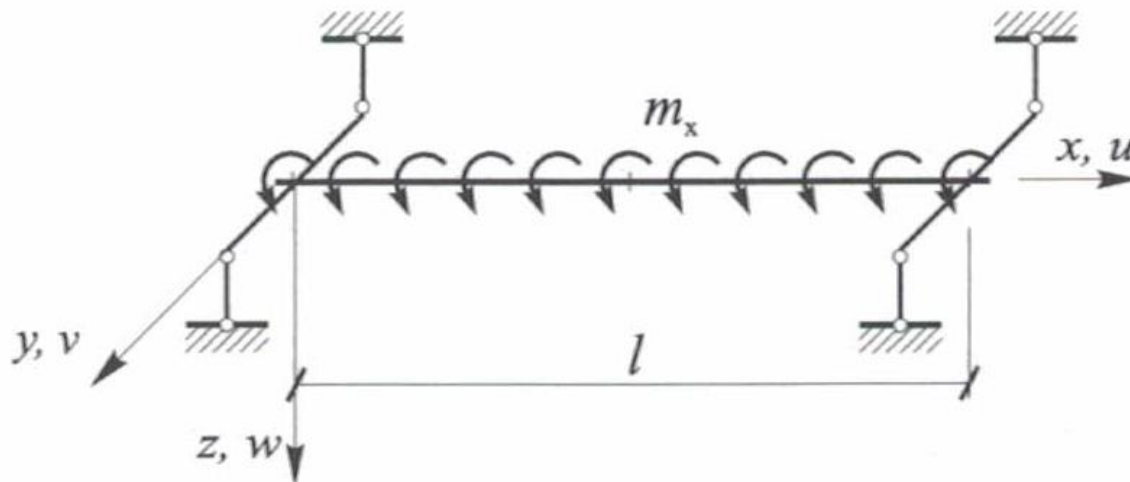
ROZWIĄZANIE OGÓLNE

Kąt skręcenia pręta obciążonego stałym momentem $m(x)$

$$\varphi(x) = \varphi_0 + \varphi_0^{(1)} x + \varphi_0^{(2)} \frac{1}{k^2} (\cosh kx - 1) + \varphi_0^{(3)} \frac{1}{k^3} (\sinh kx - kx) - \frac{m_x}{k^4 EI_\omega} \left[\frac{1}{2} (kx)^2 - \cosh kx + 1 \right]$$

Zakładając widelkowe podparcie, równanie przybiera postać

$$\varphi(x) = \frac{m_x}{k^4 EI_\omega} \left[\frac{k^2}{2} x(l-x) + \frac{\cosh k \left(\frac{l}{2} - x \right)}{\cosh \left(\frac{kl}{2} \right)} - 1 \right]$$



ROZWIĄZANIE SZCZEGÓLNE

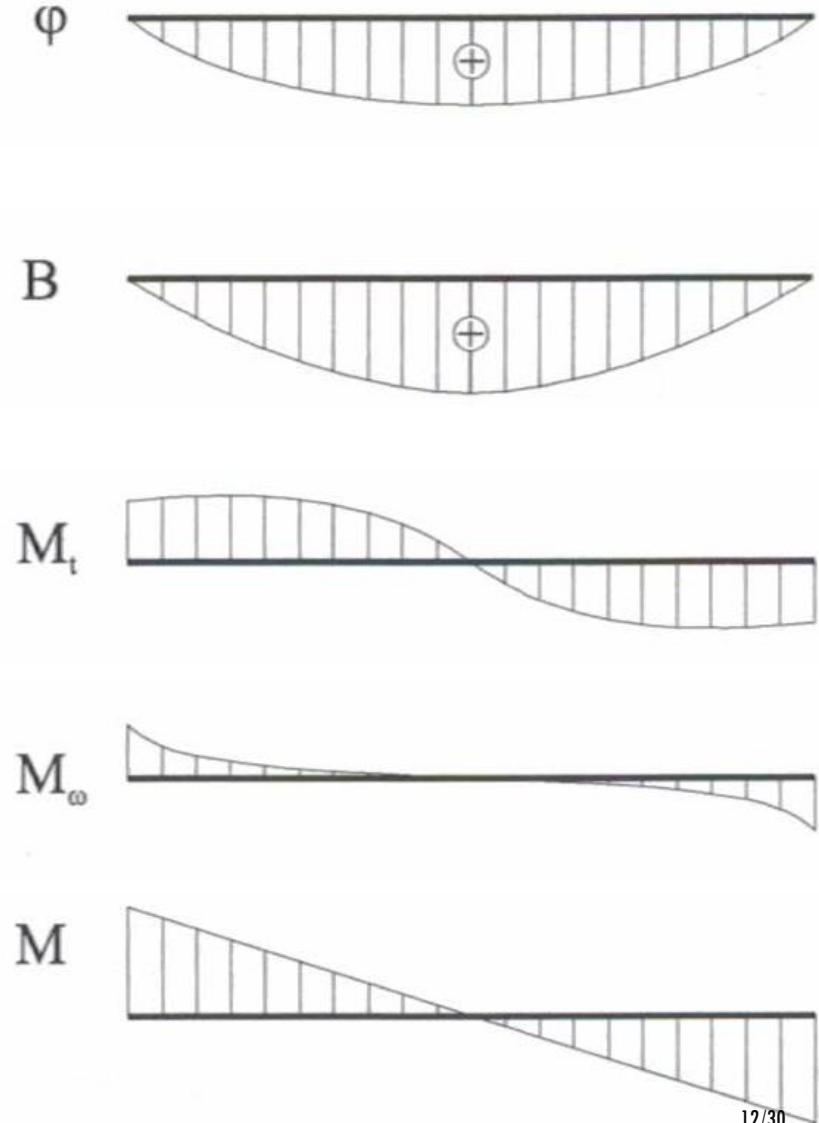
Siły przekrojowe otrzymuje się z zależności różniczkowych:

$$B(x) = \frac{m_x}{k^2} \left(1 - \frac{\cosh k \left(\frac{l}{2} - x \right)}{\cosh \left(\frac{kl}{2} \right)} \right),$$

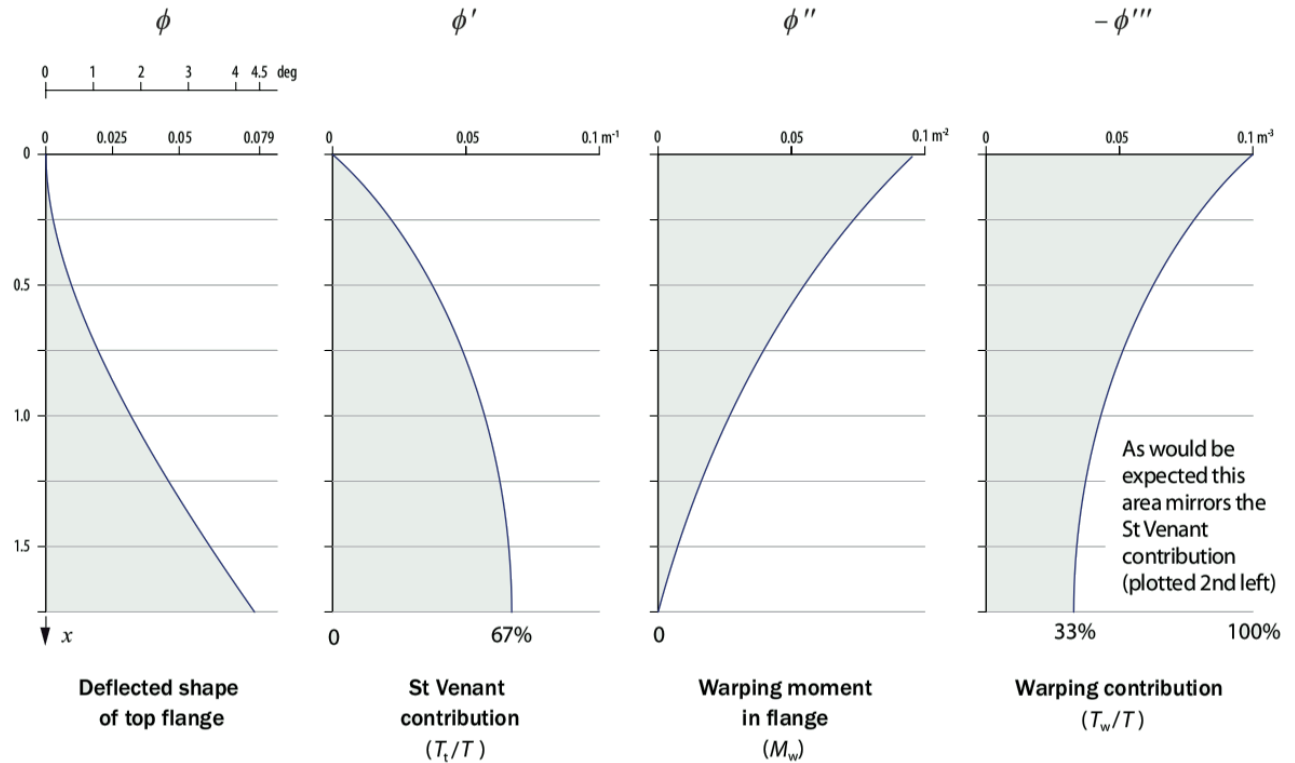
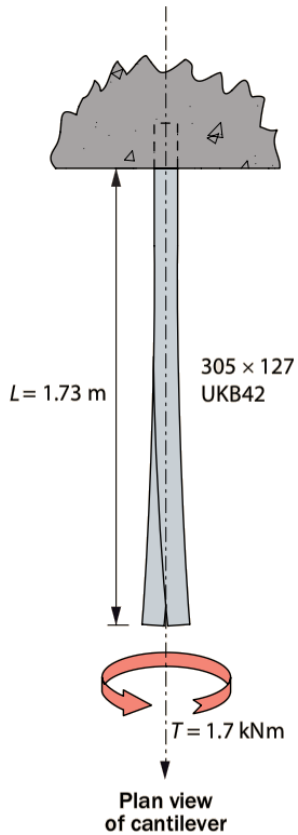
$$M_T(x) = M(x) - M_\omega(x),$$

$$M_\omega(x) = \frac{m_x}{k} \frac{\sinh k \left(\frac{l}{2} - x \right)}{\cosh \left(\frac{kl}{2} \right)},$$

$$M(x) = m_x \left(\frac{l}{2} - x \right),$$



ELEMENT WSPORNIKOWY



The expressions plotted above are:

$$\phi = \frac{Ta}{GI_T} \left[\tanh \frac{L}{a} (\cosh \frac{x}{a} - 1) - \sinh \frac{x}{a} + \frac{x}{a} \right]$$

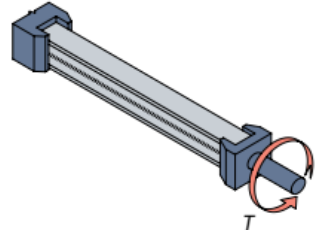
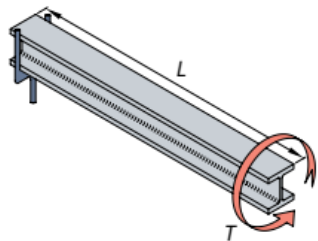
$$\phi'' = \frac{T}{GI_T a} \left[\tanh \frac{L}{a} \cosh \frac{x}{a} - \sinh \frac{x}{a} \right]$$

$$\phi' = \frac{T}{GI_T} \left[\tanh \frac{L}{a} \sinh \frac{x}{a} - \cosh \frac{x}{a} + 1 \right]$$

$$-\phi''' = \frac{T}{GI_T a^2} \left[\tanh \frac{L}{a} \sinh \frac{x}{a} - \cosh \frac{x}{a} \right]$$

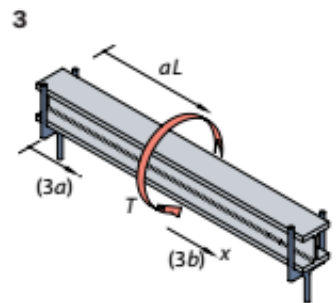
ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH

CASE	DESCRIPTION	FORMULA	GRAPH
1	<p>Torsional restraint at one end only.</p> <p>No end restraint against warping.</p> <p>Point torque T applied at free end.</p> <p><i>This case represents the torsional equivalent of a cantilever, but note that the free end may or may not be supported; torsional behaviour is no different.</i></p>	$\phi = Tx/(GI_T)$ $\phi' = T/(GI_T)$ $\phi'' = 0$ $\phi''' = 0$	<p>Not required for this simple case</p>
2	<p>Both ends restrained against warping.</p> <p>Equal and opposite torques T applied at the ends.</p>	$\phi = [Ta/(GI_T)] \{ \tanh[L/(2a)] \cosh(x/a) - \tanh[L/(2a)] + x/a - \sinh(x/a) \}$ $\phi' = [T/(GI_T)] \{ \tanh[L/(2a)] \sinh(x/a) + 1 - \cosh(x/a) \}$ $\phi'' = [T/(GI_T a)] \{ \tanh[L/(2a)] \cosh(x/a) - \sinh(x/a) \}$ $\phi''' = [T/(GI_T a^2)] \{ \tanh[L/(2a)] \sinh(x/a) - \cosh(x/a) \}$	<p>Not provided</p>



ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH

CASE	DESCRIPTION	FORMULA	GRAPH
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Torsional restraint at both ends.
No end restraint against warping.
Point torque T applied at an intermediate point αL from one end of a member of length L .

(a) for $x \leq \alpha L$

$$\phi = [Ta/(GI_T)] \{ (1 - \alpha)x/a + [\sinh(\alpha L/a)/\tanh(L/a) - \cosh(\alpha L/a)] \sinh(x/a) \}$$

$$\phi' = [T/(GI_T)] \{ (1 - \alpha) + [\sinh(\alpha L/a)/\tanh(L/a) - \cosh(\alpha L/a)] \cosh(x/a) \}$$

$$\phi'' = [T/(GI_T a)] [\sinh(\alpha L/a)/\tanh(L/a) - \cosh(\alpha L/a)] \sinh(x/a)$$

$$\phi''' = [T/(GI_T a^2)] [\sinh(\alpha L/a)/\tanh(L/a) - \cosh(\alpha L/a)] \cosh(x/a)$$

(b) for $x \geq \alpha L$

$$\phi = [Ta/(GI_T)] [(L - x)\alpha/a + \sinh(\alpha L/a) \sinh(x/a)/\tanh(L/a) - \sinh(\alpha L/a) \cosh(x/a)]$$

$$\phi' = [T/(GI_T)] [-\alpha + \sinh(\alpha L/a) \cosh(x/a)/\tanh(L/a) - \sinh(\alpha L/a) \sinh(x/a)]$$

$$\phi'' = [T/(GI_T a)] [\sinh(\alpha L/a) \sinh(x/a)/\tanh(L/a) - \sinh(\alpha L/a) \cosh(x/a)]$$

$$\phi''' = [T/(GI_T a^2)] [\sinh(\alpha L/a) \cosh(x/a)/\tanh(L/a) - \sinh(\alpha L/a) \sinh(x/a)]$$

(c) for special case of $x = \alpha = L/2$

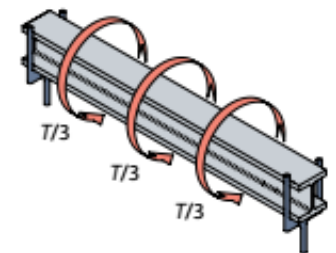
$$\phi = [Ta/(GI_T)] [L/(4a) + \{ \sinh[L/(2a)]/\tanh(L/a) - \cosh[L/(2a)] \} \sinh[L/(2a)]]$$

$$\phi' = [T/(GI_T a)] \{ \sinh[L/(2a)]/\tanh(L/a) - \cosh[L/(2a)] \} \sinh[L/(2a)]$$

For quarter and third point loading, use formulae for single point load and add together the results for the two or three applied torques (evaluated for $T/2$ or $T/3$, as appropriate).

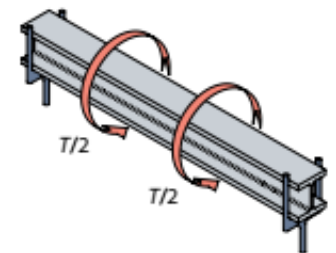
Alternatively:

Point torque $T/3$ applied at quarter points



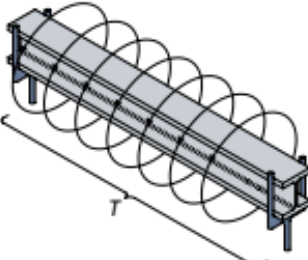
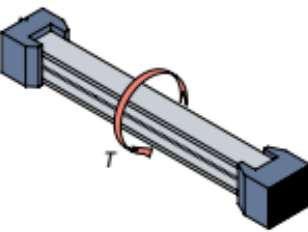
Or:

Point torque $T/2$ applied at third points.

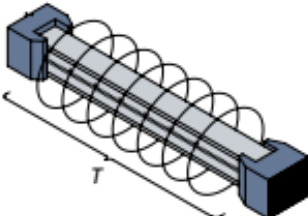
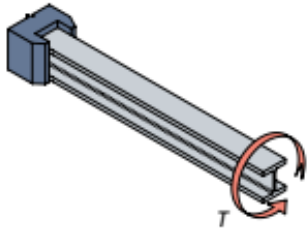
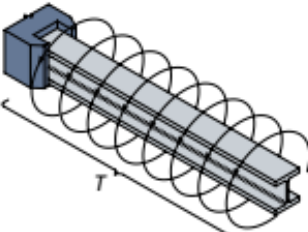


Use graphs
A (for ϕ)
B (for ϕ')

ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH

CASE	DESCRIPTION	FORMULA	GRAPH
4	<p>Torsional restraint at both ends.</p> <p>No end restraint against warping.</p> <p>Uniformly distributed torque T along full length of member.</p> 	$\phi = [Ta^2/(GI_T L)] \{ (xL - x^2)/(2a^2) + \cosh(x/a) - \tanh[L/(2a)]\sinh(x/a) - 1 \}$ $\phi' = [Ta/(GI_T L)] \{ L/(2a) - x/a + \sinh(x/a) - \tanh[L/(2a)]\cosh(x/a) \}$ $\phi'' = [T/(GI_T L)] \{ -1 + \cosh(x/a) - \tanh[L/(2a)]\sinh(x/a) \}$ $\phi''' = [T/(GI_T La)] \{ \sinh(x/a) - \tanh[L/(2a)]\cosh(x/a) \}$	<p>Use graphs C (for ϕ) D (for ϕ'')</p>
5	<p>As Case 3 but fully restrained against both warping and torsion at both ends.</p> <p><i>Note: expressions for K_1 etc are given at the foot of this table.</i></p> 	<p>(a) for $x \leq aL$</p> $\phi = [Ta/(GI_T)] \{ (K_1 - K_3 + K_4)[\cosh(x/a) - 1] - \sinh(x/a) + x/a \} / (K_1 + 1)$ $\phi' = [T/(GI_T)] \{ (K_1 - K_3 + K_4)\sinh(x/a) - \cosh(x/a) + 1 \} / (K_1 + 1)$ $\phi'' = [T/(GI_T a)] \{ (K_1 - K_3 + K_4)\cosh(x/a) - \sinh(x/a) \} / (K_1 + 1)$ $\phi''' = [T/(GI_T a^2)] \{ (K_1 - K_3 + K_4)\sinh(x/a) - \cosh(x/a) \} / (K_1 + 1)$ <p>(b) for $x \geq aL$</p> $\phi = [Ta/(GI_T)] [K_2 + K_5 \cosh(x/a) + \{ \cosh(\alpha L/a) - 1 \} / K_1 + \cosh(\alpha L/a) \} \sinh(x/a) - x/a] / (1 + 1/K_1)$ $\phi' = [T/(GI_T)] [K_2 \sinh(x/a) + \{ \cosh(\alpha L/a) - 1 \} / K_1 + \cosh(\alpha L/a) \} \cosh(x/a) - 1] / (1 + 1/K_1)$ $\phi'' = [T/(GI_T a)] [K_2 \cosh(x/a) + \{ \cosh(\alpha L/a) - 1 \} / K_1 + \cosh(\alpha L/a) \} \sinh(x/a)] / (1 + 1/K_1)$ $\phi''' = [T/(GI_T a^2)] [K_2 \sinh(x/a) + \{ \cosh(\alpha L/a) - 1 \} / K_1 + \cosh(\alpha L/a) \} \cosh(x/a)] / (1 + 1/K_1)$	<p>Not provided</p>

ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH

CASE	DESCRIPTION	FORMULA	GRAPH
6	<p>As Case 4 but fully restrained against both warping and torsion at both ends.</p> 	$\phi = [Ta/(2GI_T)] \{ [1 + \cosh(L/a)] [\cosh(x/a) - 1] / \sinh(L/a) + (1 - x/L)x/a - \sinh(x/a) \}$ $\phi' = [T/(2GI_T)] \{ [1 + \cosh(L/a)] \sinh(x/a) / \sinh(L/a) + 1 - 2x/L - \cosh(x/a) \}$ $\phi'' = [T/(2GI_T a)] \{ [1 + \cosh(L/a)] \cosh(x/a) / \sinh(L/a) - 2a/L - \sinh(x/a) \}$ $\phi''' = [T/(2GI_T a^2)] \{ [1 + \cosh(L/a)] \sinh(x/a) / \sinh(L/a) - \cosh(x/a) \}$	Not provided
7	<p>As Case 1 but one end (at $x = 0$) is restrained against warping as well as against torsion.</p> <p>Point torque T applied at free end.</p>  <p><i>Note that the free end may or may not be supported; torsional behaviour is no different.</i></p>	$\phi = [Ta/(GI_T)] \{ \tanh(L/a) [\cosh(x/a) - 1] - \sinh(x/a) + x/a \}$ $\phi' = [T/(GI_T)] [\tanh(L/a) \sinh(x/a) - \cosh(x/a) + 1]$ $\phi'' = [T/(GI_T a)] [\tanh(L/a) \cosh(x/a) - \sinh(x/a)]$ $\phi''' = [T/(GI_T a^2)] [\tanh(L/a) \sinh(x/a) - \cosh(x/a)]$	Use graph E (for both ϕ and ϕ'')
8	<p>As Case 1 but one end is restrained against both warping and torsion.</p> <p>Torque T uniformly distributed along the length of the member.</p>  <p><i>Note that the free end may or may not be supported; torsional behaviour is no different.</i></p>	$\phi = [Ta^2/(GI_T L)] \{ K_8 [\cosh(x/a) - 1] - (L/a) \sinh(x/a) + (x/a) [L/a - x/(2a)] \}$ $\phi' = [Ta/(GI_T L)] [K_8 \sinh(x/a) - (L/a) \cosh(x/a) + (L/a - x/a)]$ $\phi'' = [T/(GI_T L)] [K_8 \cosh(x/a) - (L/a) \sinh(x/a) - 1]$ $\phi''' = [T/(GI_T a L)] [K_8 \sinh(x/a) - (L/a) \cosh(x/a)]$	Use graph E (for both ϕ and ϕ'')

ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH

CASE	DESCRIPTION	FORMULA	GRAPH
9	As Case 4 but one end is fully restrained against both warping and torsion.	$\phi = [Ta^2/(GI_T L)] \{K_{10}[\tanh(L/a) - x/a - \tanh(L/a)\cosh(x/a) + \sinh(x/a)] + \cosh(x/a)/\cosh(L/a) - 1/\cosh(L/a) - x^2/(2a^2)\}$ $\phi' = [Ta/(GI_T L)] \{K_{10}[-1 - \tanh(L/a)\sinh(x/a) + \cosh(x/a)] + \sinh(x/a)/\cosh(L/a) - x/a\}$ $\phi'' = [T/(GI_T L)] \{K_{10}[-\tanh(L/a)\cosh(x/a) + \sinh(x/a)] + \cosh(x/a)/\cosh(L/a) - 1\}$ $\phi''' = [T/(GI_T aL)] \{K_{10}[-\tanh(L/a)\sinh(x/a) + \cosh(x/a)] + \sinh(x/a)/\cosh(L/a)\}$	Not provided
10	Torsional restraints at both ends. No end restraint against warping. Torque per unit length varying from zero at LH end ($x = 0$) to $2T/L$ at RH end ($x = L$). Total torque T .	$\phi = [2T/(GI_T)] [x/6 - xa^2/L^2 + (a^2/L)\sinh(x/a)/\sinh(L/a) - x^3/(6L^2)]$ $\phi' = [2T/(GI_T)] [1/6 - a^2/L^2 + (a/L)\cosh(x/a)/\sinh(L/a) - x^2/(2L^2)]$ $\phi'' = [2T/(GI_T)] [(1/L)\sinh(x/a)/\sinh(L/a) - x/L^2]$ $\phi''' = [2T/(GI_T)] \{[1/(aL)]\cosh(x/a)/\sinh(L/a) - 1/L^2\}$	Not provided

$$K_1 = \{[1 - \cosh(\alpha L/a)]/\tanh(L/a) + [\cosh(\alpha L/a) - 1]/\sinh(L/a) + \sinh(\alpha L/a) - \alpha L/a\} / \{[\cosh(L/a) + \cosh(\alpha L/a)\cosh(L/a) - \cosh(\alpha L/a) - 1]/\sinh(L/a) + (\alpha - 1)L/a - \sinh(\alpha L/a)\}$$

$$K_2 = [\cosh(\alpha L/a) - 1]/[K_1 \sinh(L/a)] + [\cosh(\alpha L/a) - \cosh(L/a) + (L/a)\sinh(L/a)]/\sinh(L/a)$$

$$K_3 = 1/\sinh(L/a) + \sinh(\alpha L/a) - \cosh(\alpha L/a)]/\tanh(L/a)$$

$$K_4 = \sinh(\alpha L/a) - \cosh(\alpha L/a)]/\tanh(L/a) + 1/\tanh(L/a)$$

$$K_5 = [1 - \cosh(\alpha L/a)]/[K_1 \tanh(L/a)] + [1 - \cosh(\alpha L/a)\cosh(L/a)]/\sinh(L/a)$$

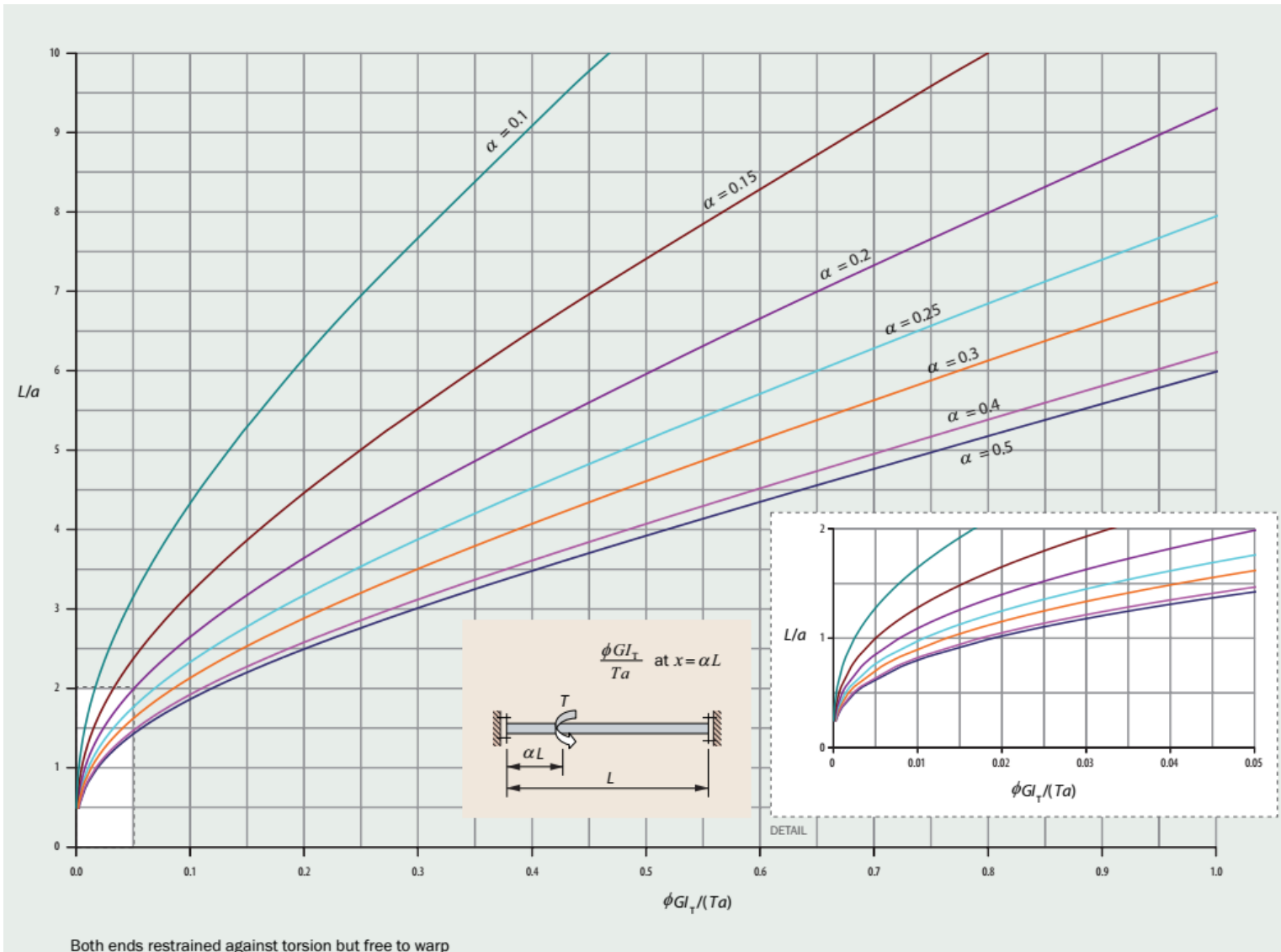
$$K_8 = \tanh(L/a)[L/a - \sinh(L/a)] + \cosh(L/a)$$

$$K_{10} = [L^2/(2a^2) - 1 + 1/\cosh(L/a)]/[\tanh(L/a) - (L/a)]$$

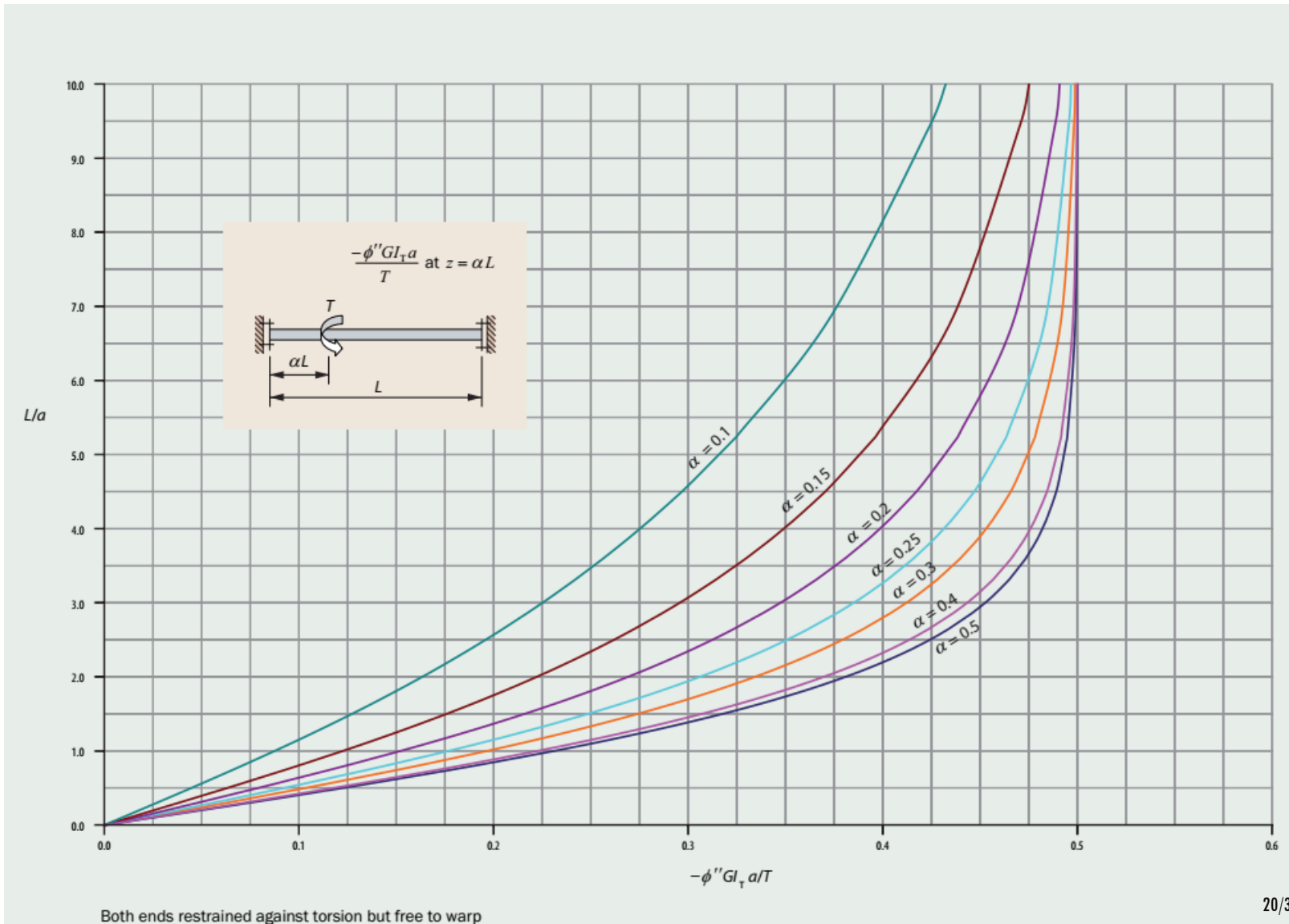
Caution: Note the comments in Sections 2.6 and 7.6 before assuming warping fixity at member ends (indicated by the vice-like device).

In every case, T is the total torque acting on the member.

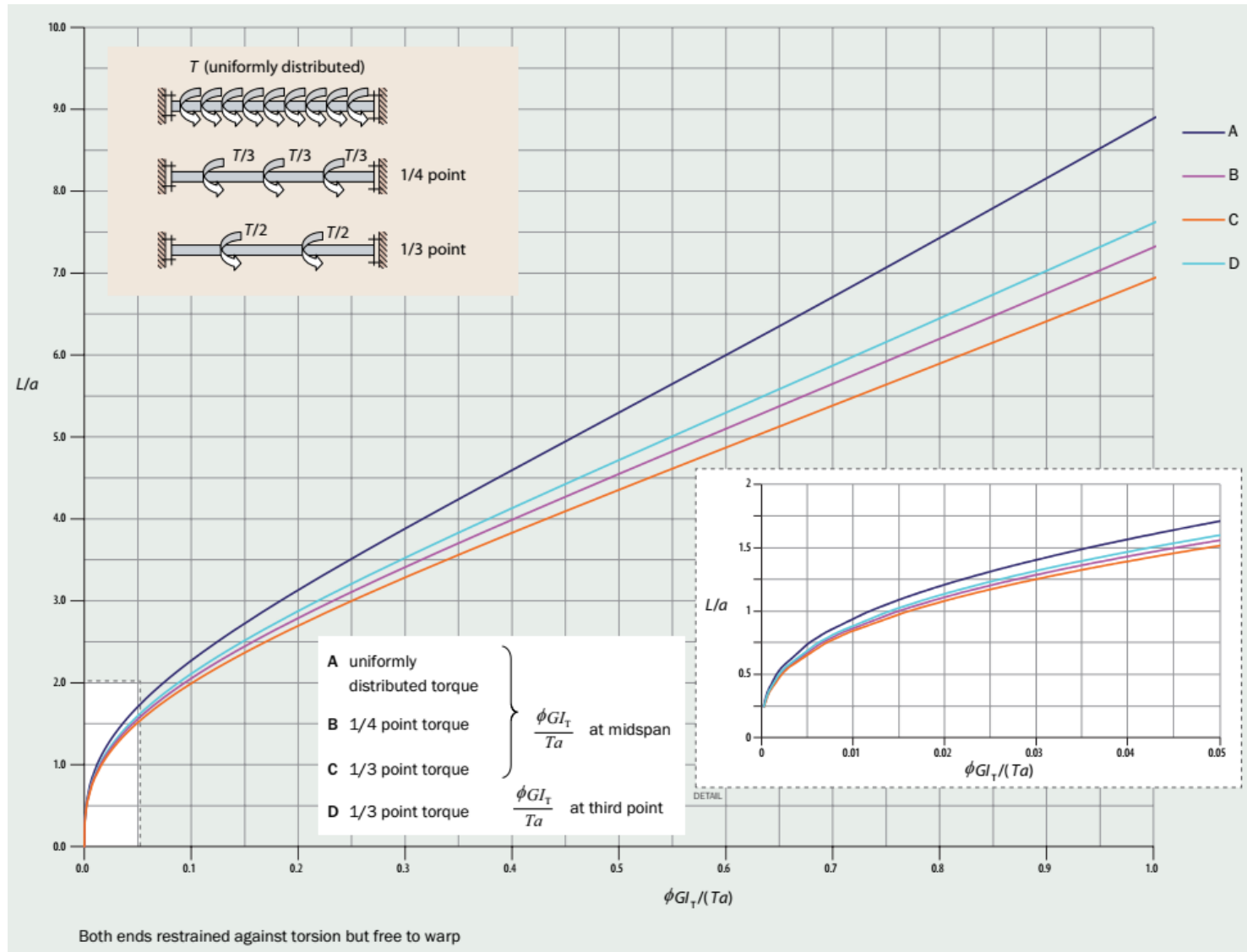
ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH



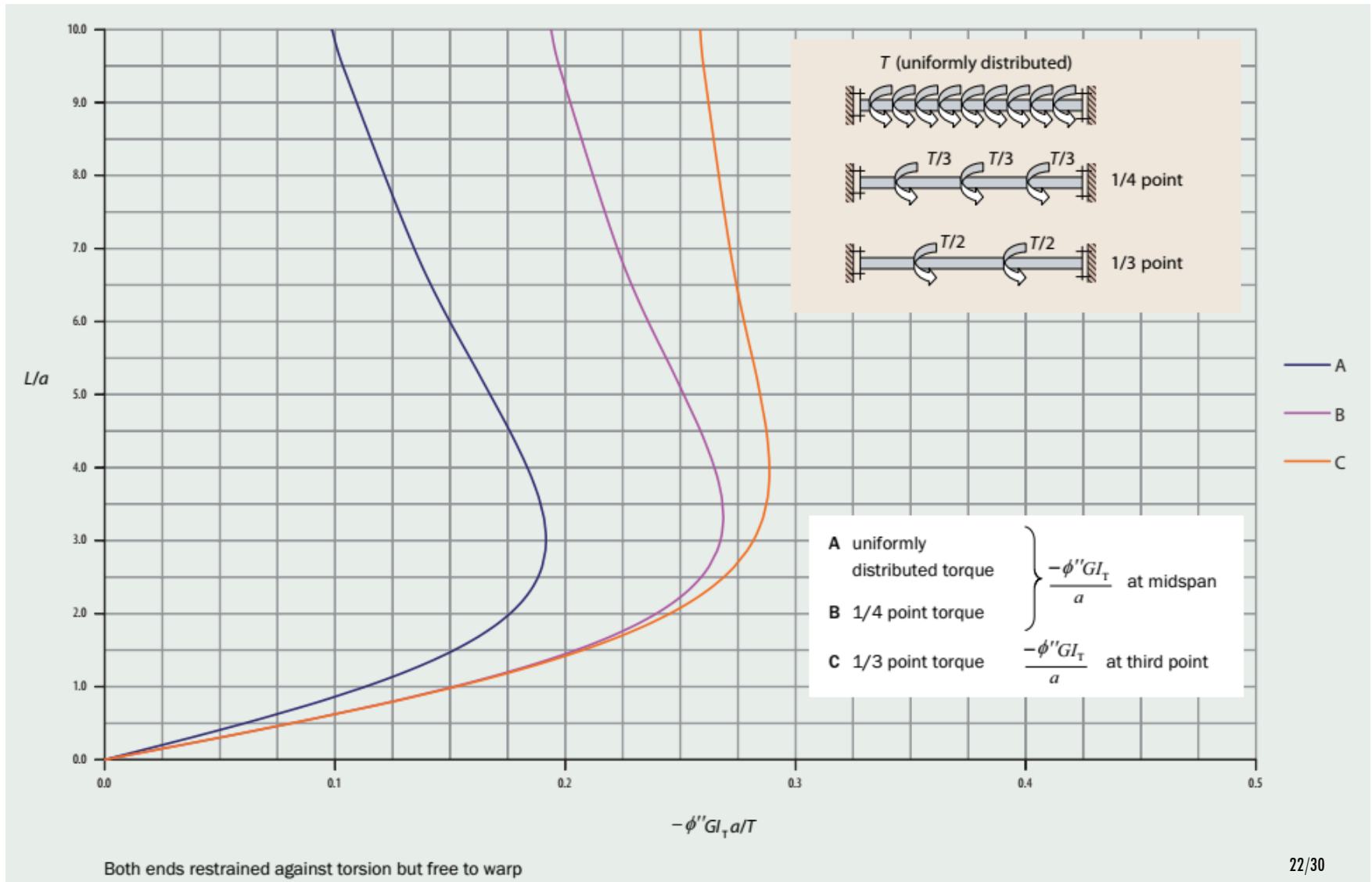
ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH



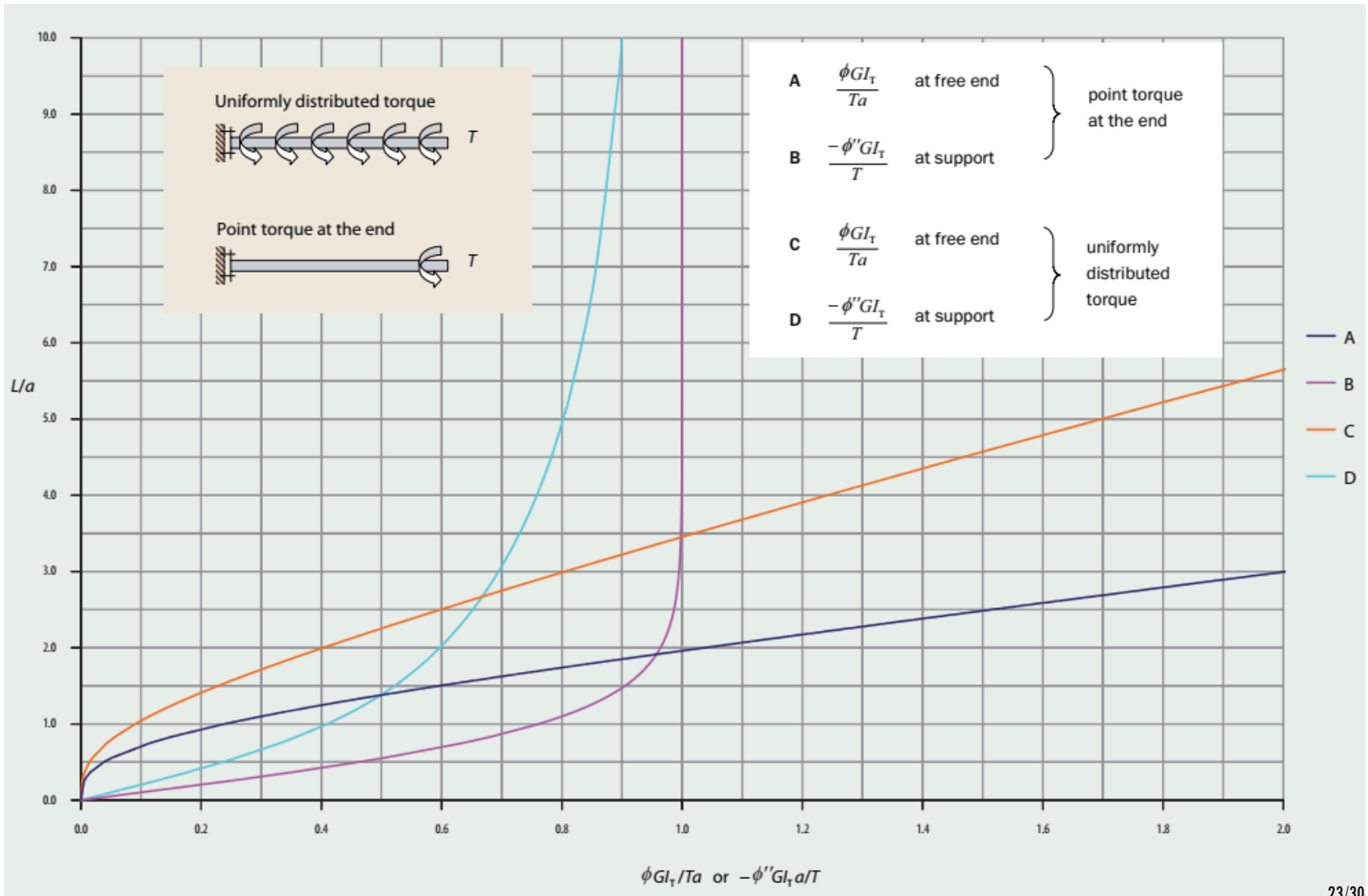
ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH



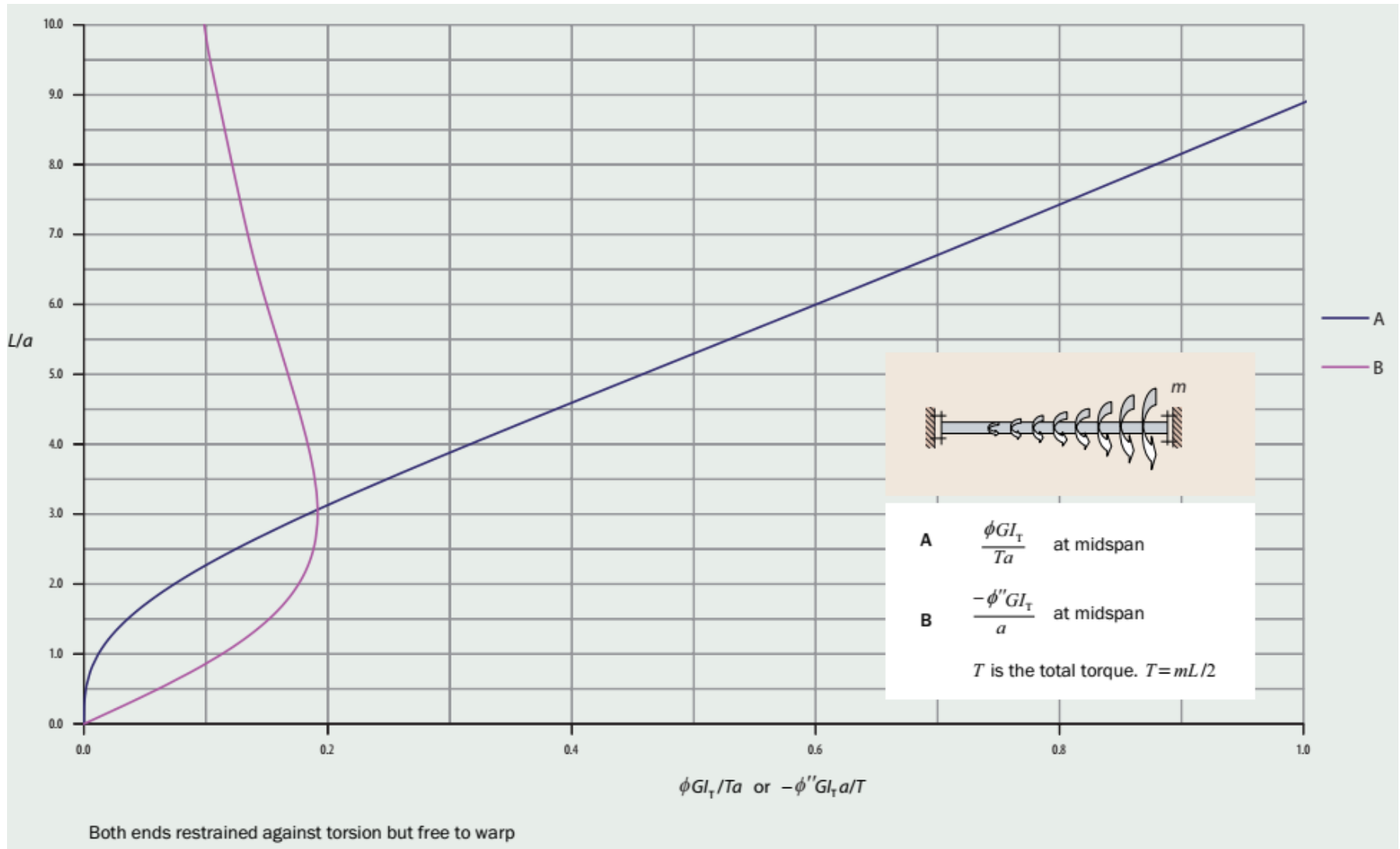
ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH



ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH



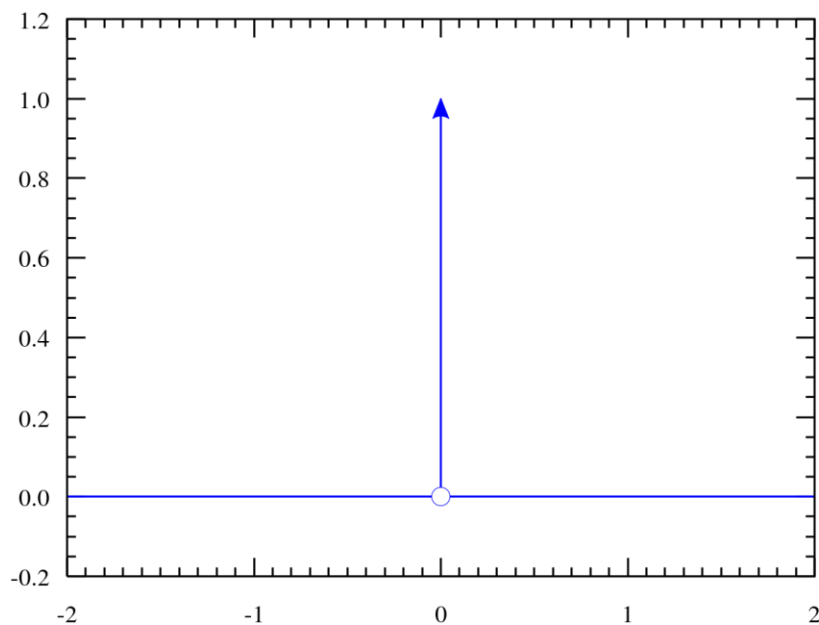
ROZWIĄZANIA PRZYPADKÓW SZCZEGÓLNYCH



DYSTRYBUCJE DIRACA I HEAVISIDEA

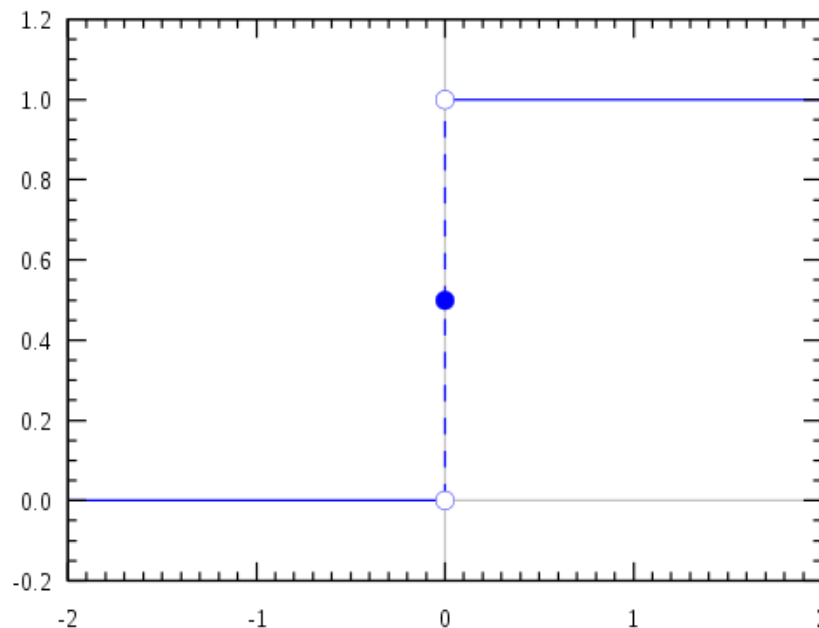
Delta Diraca

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$



Funkcja Heavisidea

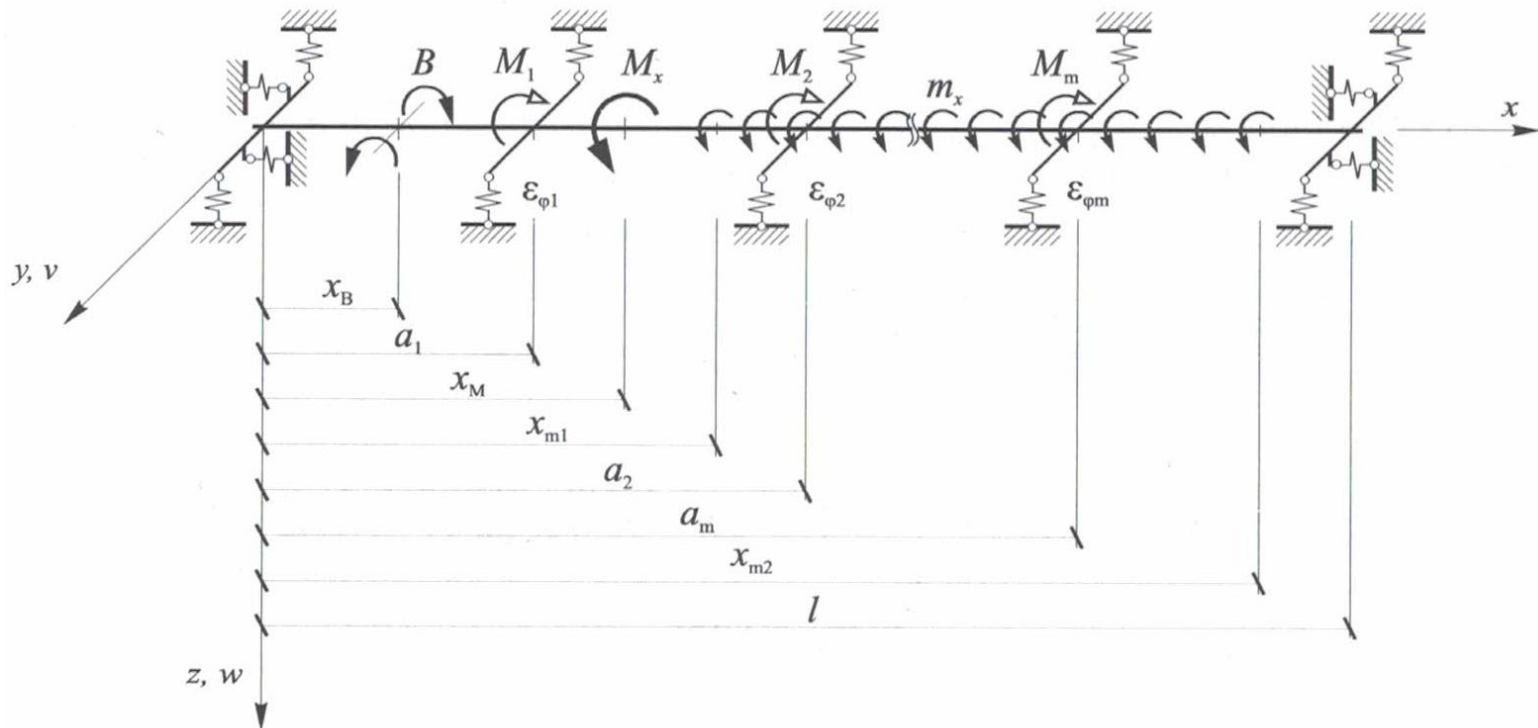
$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$



ELEMENTY WIELOPRZĘSŁOWE

Równanie różniczkowe elementu wieloprzęsłowego:

$$\begin{aligned}
 EI_{\omega} \varphi^{(4)}(x) - GI_T \varphi^{(2)}(x) &= \\
 &= B \delta^{(1)}(x - x_B) + M_x \delta(x - x_B) - \sum_{p=1}^m M_p \delta(x - a_p) \\
 &+ m_x (H(x - x_{m1}) - H(x - x_{m2}))
 \end{aligned}$$



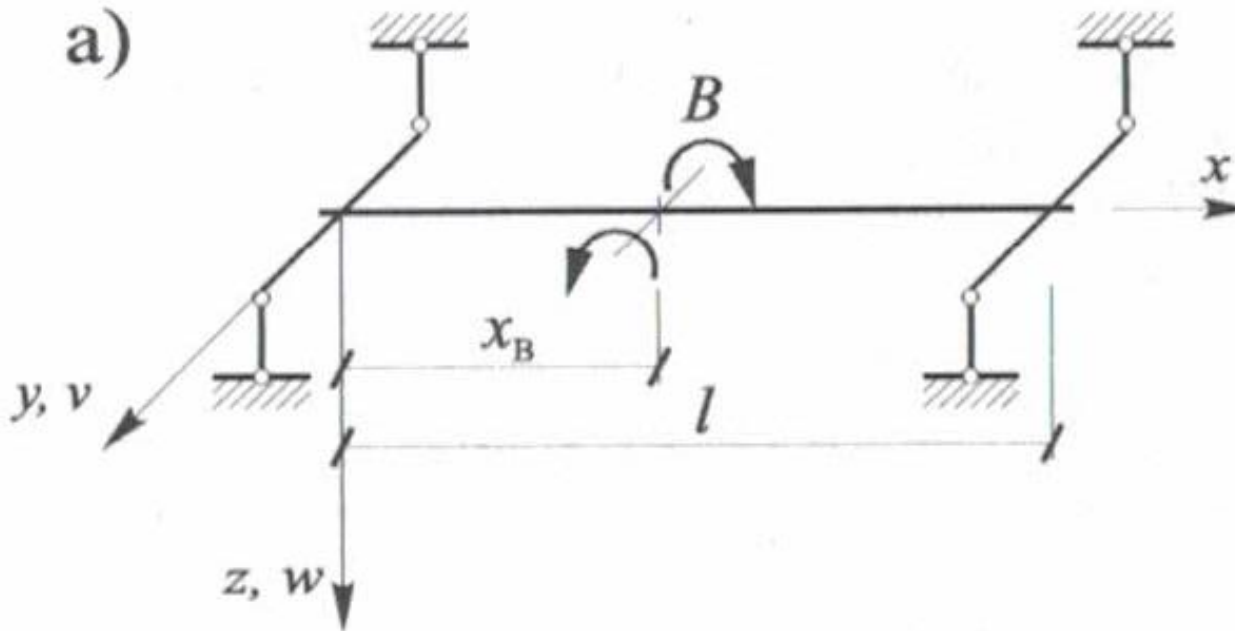
ROZWIĄZANIE OGÓLNE

Rozwiązanie ogólne r-nia skręcanego pręta wieloprzęstowego:

$$\begin{aligned}\varphi(x) = & \varphi_0 + \varphi_0^{(1)} x + \varphi_0^{(2)} \frac{1}{k^2} (\cosh kx - 1) + \\ & + \varphi_0^{(3)} \frac{1}{k^3} (\sinh kx - kx) + \frac{B}{EI_\omega} \frac{1}{k^2} [\cosh k(x - x_B) - 1] H(x - x_B) \\ & + \frac{M_x}{EI_\omega} \frac{1}{k^3} [\sinh k(x - x_M) - k(x - x_M)] H(x - x_M) \\ & - \sum_{p=1} \frac{M_p}{EI_\omega} \frac{1}{k^3} [\sinh k(x - a_p) - k[x - a_p] H(x - a_p)] \\ & + \frac{m_x}{EI_\omega} \frac{1}{2k^4} \{ [2 \cosh k(x - x_{m1}) - k^2(x - x_{m1})^2 - 2] H(x - x_{m1}) \\ & - [2 \cosh k(x - x_{m2}) - k^2(x - x_{m2})^2 - 2] H(x - x_{m2}) \}\end{aligned}$$

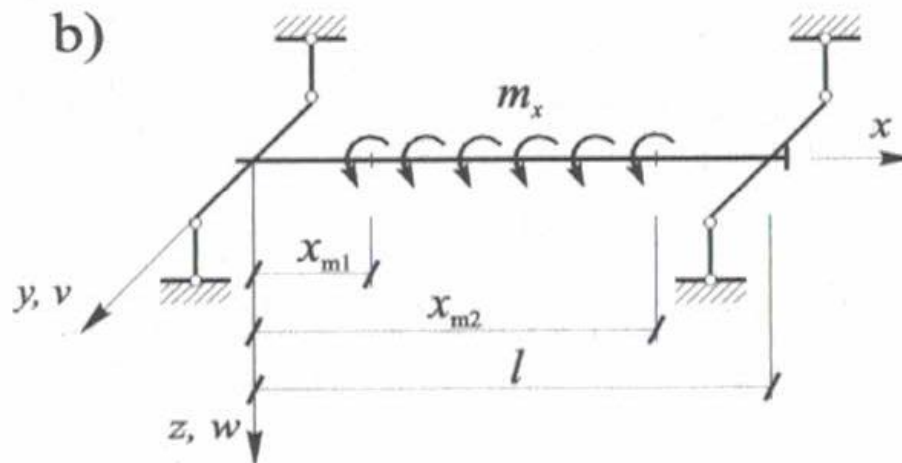
PRZYPADKI SZCZEGÓLNE

$$\varphi(x) = \frac{B}{k^2 EI_\omega} \left\{ \frac{x}{l} - \frac{\cosh k(l - x_B)}{\sinh kl} \sinh kx + [\cosh k(x - x_B) - 1] H(x - x_B) \right\}$$



PRZYPADKI SZCZEGÓLNE

$$\begin{aligned} \varphi(x) = & \\ = & \frac{m_x}{k^4 EI_\omega} \left\{ -(x_{m1} - x_{m2})k^2 x \right. \\ & \frac{[\sinh k(l - x_{m1}) - \sinh k(l - x_{m2})] \sinh kl - \left[\cosh k(l - x_{m1}) - \cosh k(l - x_{m2}) - \frac{k^2}{2}(x_{m1}^2 - x_{m2}^2) \right] \cosh kl}{\sinh kl - kl \cosh kl} \cosh kl \\ & + \frac{kl[\sinh kl(l - x_{m1}) - \sinh k(l - x_{m2})] - \cosh k(l - x_{m1}) \cosh k(l - x_{m2}) + \frac{k^2}{2}(x_{m1}^2 - x_{m2}^2)}{\sinh kl - kl \cosh kl} \sinh kx \\ & \left. + \left[\cosh k(x - x_{m1}) - \frac{k^2}{2}(x - x_{m1})^2 - 1 \right] H(x - x_{m1}) - \left[\cosh k(x - x_{m2}) - \frac{k^2}{2}(x - x_{m2})^2 - 1 \right] H(x - x_{m2}) \right\} \end{aligned}$$



PRZYPADKI SZCZEGÓLNE

$$\varphi(x) = \frac{M_x}{k^3 EI_\omega} \left\{ kx - \frac{\sinh kl \cosh k(l - x_M) - [\sinh k(l - x_M) + kx_M] \cosh kl}{\sinh kl - kl \cosh kl} kx \right.$$

$$- \frac{\sinh k(l - x_M) - kl \cosh k(l - x_M) + kx_M}{\sinh kl - kl \cosh kl} \sinh kx$$

$$\left. + [\sinh k(x - x_M) - k(x - x_M) - k(x - x_M)]H(x - x_M) \right\}$$

